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Modeling the coastal ocean over a time period of several weeks

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ABSTRACT

From a scale analysis of hydrodynamic phenomena having a significant action on the drift of an object in coastal ocean waters, we deduce equations modeling the associated hydrodynamic fields over a time period of several weeks. These models are essentially non linear hyperbolic systems of PDE involving a small parameter. Then from the models we extract a simplified and nevertheless typical one for which we prove that its classical solution exists on a time interval which is independent of the small parameter. We then show that the solution weak-$*$ converges as the small parameter goes to zero and we characterize the equation satisfied by the weak-$*$ limit.

1. Introduction

This paper is part of a work program concerning the modeling of object drift in near coastal ocean waters over a several week time period.

The final target of this program is to develop methods to forecast the drift of things like containers, lost objects or oil spill over long periods of time in near coastal ocean areas. Such methods would be of interest for services in charge of maritime safety, environmental studies or pollution impact assessment. To reach this target, several research topics need to be further investigated. For instance, improvements are needed in the field of the numerical methods to simulate long term drift, in the modeling and simulation of the near coastal ocean waters, in the understanding of ocean-object and

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ocean-spill interactions and, of course, in the integration of all those aspects to move toward a coherent theory.

In the previous paper of this work program, Ailliot, Frénod and Monbet [2] considered the numerical method facet. We built a numerical method coupling a two scale expansion method, explored in [10], and a stochastic wind simulator, in the spirit of [1,3,25], in order to estimate probability of events that may happen to the considered object, such as running aground in a given area. In [2], the simplified model was supposed to describe the object dynamic in the ocean. It involved an ocean velocity field which was decomposed into a sum of a velocity due to the tide wave and of a perturbation. Both of them were periodic of the tide period and with modulated amplitude, and the fields used for the numerical simulations were not realistic.

The present paper deals with the modeling of near coastal ocean. The sea velocity and the fluctuation of the sea level due to the tide wave are well known in many coastal areas of the world. Those main fields are perturbed by fields with a smaller order of magnitude having a net long-term result. Such perturbations, which are produced by meteorological factors, propagate and interact with the main fields. The precise aim of the present paper is to make a first step toward the set up of a modeling procedure in order to establish partial differential equation systems describing the evolution of those perturbations and to suggest ideas to solve them.

The paper is organized as follows. In Section 2, we summarize the main mathematical results.

Then, in Section 3, we set up the previously evoked modeling procedure. The models which are built in this section are deduced from the Shallow Water Equations via a scale analysis of the geophysical phenomena concerned and of the geometrical size of the concerned domain. As for the size of the coastal domain and the characteristic order of magnitude of the wind velocity, we consider several possibilities giving rise to several models. They are all essentially hyperbolic systems of partial differential equations with a singular perturbation involving a small parameter.

From those models, in Section 4, we extract a simplified one which is nevertheless typical. For it, adapting classical methods for hyperbolic systems (see Kato [18], Majda [23], Klainerman and Majda [19,20], Schochet [33,34], and Métivier and Schochet [24]), we prove that its classical solution exits with a time existence independent of the small parameter.

In Section 5, using a homogenization method (see Tartar [37], Bensoussan, Lions and Papanicolaou [6], Sanchez-Palencia [31] and Lions [21]), we set that this classical solution weak-$*$ converges to a function. Keeping within the mind frame of Frénod [9], Frénod and Hamdache [11], Frénod, Raviart and Sonnendrücker [12], Joly, Métivier and Rauch [17] or Schochet [35] we finally look for the form of this function and establish the equations allowing for its computation.

Finally, in Section 6 we conclude and give some perspectives.

2. Results

In this section we present the main results. We first present one of the models, involving a small parameter, set out in this paper. Then we state a theorem claiming the existence of the solution to a simplified version of this model. Finally, we exhibit the asymptotic behavior of this solution as the small parameter goes to zero.

The model we present now, and which is set out among others in Section 3, describes the evolution, over a several months time period, of the perturbation of the sea velocity and of the sea level in an ocean domain above the continental shelf at a latitude about 45° and with stormy weather conditions.

The small parameter involved in this model is the ratio tide duration on observation time scale. The first one is about 13 hours and the second is about three months. Hence the involved small parameter is:

$$\varepsilon = \frac{1}{200}. \tag{2.1}$$

Variables and fields involved are all rescaled; rescaled meaning that the order of magnitude of those variables and fields is one and that they have no physical dimension.

The rescaled velocity of the sea $\tilde{\mathbf{M}}$ and the water depth $\tilde{\mathcal{H}}$ induced by the tide wave are considered as known and periodic with modulated amplitude. In other words, t being the rescaled time and $\mathbf{x}$ the rescaled position,

$$\tilde{\mathbf{M}}(t,\mathbf{x}) = \mathbf{M}\left(t, \frac{t}{\varepsilon}, \mathbf{x}\right) \quad \text{and} \quad \tilde{\mathcal{H}}(t,\mathbf{x}) = \mathcal{H}\left(t, \frac{t}{\varepsilon}, \mathbf{x}\right), \tag{2.2}$$

where $\mathbf{M}$ and $\mathcal{H}$ are regular functions and where $\theta \mapsto (\mathbf{M}(t,\theta,\mathbf{x}), \mathcal{H}(t,\theta,\mathbf{x}))$ is 1-periodic.

The model says that the total sea velocity, expressed in km/h, writes $0.5(\tilde{\mathbf{M}} + \varepsilon\tilde{\mathbf{N}})$ and that the total sea level is $\frac{3}{2\varepsilon}(E + 2\varepsilon\tilde{\mathcal{H}} + 2\varepsilon^2\tilde{\mathcal{I}})$, where E is the rescaled mean sea level and where $\tilde{\mathbf{N}}$ and $\tilde{\mathcal{I}}$ are rescaled perturbations. Moreover, $(\tilde{\mathcal{I}}, \tilde{\mathbf{N}})$ is solution to

$$\begin{aligned}
&\frac{\partial \tilde{\mathcal{I}}}{\partial t} + \nabla\left(\frac{1}{\varepsilon}E + 2\tilde{\mathcal{H}}\right)\cdot\tilde{\mathbf{N}} + \left(\frac{1}{\varepsilon}E + 2\tilde{\mathcal{H}}\right)\nabla\cdot\tilde{\mathbf{N}} + 2(\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{M}} + 2\tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{M}}) \\
&\qquad + 2\varepsilon\big((\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{N}} + \tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{N}})\big) = 0, \qquad (2.3) \\
&\frac{\partial \tilde{\mathbf{N}}}{\partial t} + 2(\nabla\tilde{\mathbf{N}})\tilde{\mathbf{M}} + 2(\nabla\tilde{\mathbf{M}})\tilde{\mathbf{N}} + 2\varepsilon(\nabla\tilde{\mathbf{N}})\tilde{\mathbf{N}} + \frac{\pi}{2\varepsilon}\tilde{\mathbf{N}}^{\perp} + \frac{1}{4\varepsilon}\nabla\tilde{\mathcal{I}} - 13\varepsilon^4\Delta\tilde{\mathbf{M}} - 13\varepsilon^5\Delta\tilde{\mathbf{N}} \\
&\qquad - 13\varepsilon^4\frac{(\nabla\tilde{\mathbf{M}})\nabla(E+2\varepsilon\tilde{\mathcal{H}})}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}} - 26\varepsilon^6\frac{(\nabla\tilde{\mathbf{M}})\nabla\tilde{\mathcal{I}}}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}} - 13\varepsilon^5\frac{(\nabla\tilde{\mathbf{N}})\nabla(E+2\varepsilon\tilde{\mathcal{H}})}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}} \\
&\qquad - 26\varepsilon^7\frac{(\nabla\tilde{\mathbf{N}})\nabla\tilde{\mathcal{I}}}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}} + \frac{3}{\varepsilon}\frac{\sqrt{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}}{1+\frac{0.8}{\varepsilon^2}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}\tilde{\mathbf{M}} \\
&\qquad + 3\frac{\sqrt{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}}{1+\frac{0.8}{\varepsilon^2}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}\tilde{\mathbf{N}} \\
&\quad = 6\frac{\sqrt{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}}{1+\frac{1.5}{\varepsilon}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}\left(\frac{1}{\varepsilon}\tilde{\mathbf{W}} - \tilde{\mathbf{M}}\right) - 6\varepsilon\frac{\sqrt{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}}{1+\frac{1.5}{\varepsilon}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}\tilde{\mathbf{N}}. \qquad (2.4)
\end{aligned}$$

In this system, $\tilde{\mathbf{W}}$ is the rescaled wind velocity, $\tilde{\mathbf{N}}^{\perp} = (-\tilde{\mathbf{N}}_2, \tilde{\mathbf{N}}_1)$, Δ stands for the Laplacian, $\nabla\cdot$ for the divergence operator and ∇ stands for the gradient of scalar fields and for the Jacobian matrix of bi-dimensional fields.

Motivated by this system, we consider a simplified version of it which consists in considering that the ocean bottom is flat, i.e. $E \equiv 1$, in forgetting all the power of ε greater than 1 and in setting all constants to 1:

$$\frac{\partial \tilde{\mathcal{I}}}{\partial t} + (\nabla\tilde{\mathcal{H}})\cdot\tilde{\mathbf{N}} + \left(\frac{1}{\varepsilon} + \tilde{\mathcal{H}}\right)(\nabla\cdot\tilde{\mathbf{N}}) + (\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{M}} + \tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{M}}) + \varepsilon\big((\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{N}} + \tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{N}})\big) = 0, \tag{2.5}$$

$$\frac{\partial \tilde{\mathbf{N}}}{\partial t} + (\nabla\tilde{\mathbf{N}})\tilde{\mathbf{M}} + (\nabla\tilde{\mathbf{M}})\tilde{\mathbf{N}} + \varepsilon(\nabla\tilde{\mathbf{N}})\tilde{\mathbf{N}} + \frac{1}{\varepsilon}\tilde{\mathbf{N}}^{\perp} + \frac{1}{\varepsilon}\nabla\tilde{\mathcal{I}} = \tilde{\mathbf{W}}. \tag{2.6}$$

In this system, $t \in [0, T]$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. The unknowns are $\tilde{\mathcal{I}} \equiv \tilde{\mathcal{I}}(t,\mathbf{x})$ and $\tilde{\mathbf{N}} \equiv \tilde{\mathbf{N}}(t,\mathbf{x})$. Their evolution is influenced by $\tilde{\mathbf{M}}$ and $\tilde{\mathcal{H}}$ for which we assume (2.2) and by $\tilde{\mathbf{W}}$ for which we also assume

$$\tilde{\mathbf{W}}(t,\mathbf{x}) = \mathbf{W}\left(t, \frac{t}{\varepsilon}, \mathbf{x}\right), \tag{2.7}$$

with function $\mathbf{W}$ regular and with $\theta \mapsto \mathbf{W}(t,\theta,\mathbf{x})$ being 1-periodic. This assumption is not really convenient for real wind velocity time series but is comfortable from a mathematical point of view (see Ailliot, Frénod and Monbet [2] for a more detailed discussion). Moreover, we equip this system with the following initial conditions

$$\tilde{\mathcal{I}}_{|t=0} = \tilde{\mathcal{I}}_0, \qquad \tilde{\mathbf{N}}_{|t=0} = \tilde{\mathbf{N}}_0, \tag{2.8}$$

and we can claim the following theorem.

Theorem 2.1. *Under assumptions* (2.2) *and* (2.7), *if* $(\tilde{\mathcal{I}}_0, \tilde{\mathbf{N}}_0) \in (H^s(\mathbb{R}^2))^3$ *with* $s > 3$, *then there exists a time* T, *not depending on* ε, *such that the classical solution* $(\tilde{\mathcal{I}}, \tilde{\mathbf{N}}) \in (C([0,T], (H^s(\mathbb{R}^2))^3) \cap C^1([0,T], (H^{s-1}(\mathbb{R}^2))^3))$ *of* (2.5), (2.6) *and* (2.8) *exits and is unique. Moreover this solution satisfies*

$$\sup_{t\in[0,T]} \big\|(\tilde{\mathcal{I}}, \tilde{\mathbf{N}})\big\|_s \leqslant c, \tag{2.9}$$

for a constant c *not depending on* ε, *where* $\|\ \|_s$ *stands for the norm in* $(H^s(\mathbb{R}^2))^3$.

Concerning the asymptotic behavior of $(\tilde{\mathcal{I}}, \tilde{\mathbf{N}})$, as ε goes to zero, we have the following result.

Theorem 2.2. *Under the assumptions of Theorem* 2.1, *there exist functions* $\mathcal{I} \equiv \mathcal{I}(t,\mathbf{x}) \in C([0,T], H^s(\mathbb{R}^2))$ *and* $\mathbf{N} \equiv \mathbf{N}(t,\mathbf{x}) \in C([0,T], (H^s(\mathbb{R}^2))^2)$, *such that as* ε *goes to* 0, *the solution* $(\tilde{\mathcal{I}}, \tilde{\mathbf{N}})$ *of* (2.5), (2.6) *and* (2.8) *weak-∗ converges to* $(\mathcal{I}, \mathbf{N})$ *in* $L^\infty([0,T], (H^s(\mathbb{R}^2))^3)$. *Moreover,* $\mathcal{I}$ *and* $\mathbf{N}$ *are linked by*

$$\mathbf{N}_1(t,\mathbf{x}) = -\frac{\partial \mathcal{I}}{\partial x_2}(t,\mathbf{x}), \qquad \mathbf{N}_2(t,\mathbf{x}) = \frac{\partial \mathcal{I}}{\partial x_1}(t,\mathbf{x}), \tag{2.10}$$

and $\mathcal{I}$ *is solution to*

$$\begin{aligned}
&\frac{\partial(\mathcal{I} - \Delta\mathcal{I})}{\partial t} + \overline{\overline{\mathbf{M}}} \cdot \nabla\mathcal{I} - \frac{\partial\big(\overline{\overline{\mathbf{M}_1}}\frac{\partial^2\mathcal{I}}{\partial x_1^2}\big)}{\partial x_1} - \frac{\partial\big(\overline{\overline{\mathbf{M}_2}}\frac{\partial^2\mathcal{I}}{\partial x_1\partial x_2}\big)}{\partial x_1} - \frac{\partial\big(\overline{\overline{\mathbf{M}_1}}\frac{\partial^2\mathcal{I}}{\partial x_1\partial x_2}\big)}{\partial x_2} - \frac{\partial\big(\overline{\overline{\mathbf{M}_2}}\frac{\partial^2\mathcal{I}}{\partial x_2^2}\big)}{\partial x_2} \\
&\quad - \big(\nabla\overline{\overline{\mathcal{H}}}\big)^\perp \cdot \nabla\mathcal{I} + \big(\nabla \cdot \overline{\overline{\mathbf{M}}}\big)\mathcal{I} \\
&\quad + \frac{\partial\big(\frac{\partial\overline{\overline{\mathbf{M}_2}}}{\partial x_1}\frac{\partial\mathcal{I}}{\partial x_2}\big)}{\partial x_1} - \frac{\partial\big(\frac{\partial\overline{\overline{\mathbf{M}_2}}}{\partial x_2}\frac{\partial\mathcal{I}}{\partial x_1}\big)}{\partial x_1} - \frac{\partial\big(\frac{\partial\overline{\overline{\mathbf{M}_1}}}{\partial x_1}\frac{\partial\mathcal{I}}{\partial x_2}\big)}{\partial x_2} + \frac{\partial\big(\frac{\partial\overline{\overline{\mathbf{M}_1}}}{\partial x_2}\frac{\partial\mathcal{I}}{\partial x_1}\big)}{\partial x_2} \\
&= \frac{\partial\overline{\overline{\mathbf{W}_1}}}{\partial x_2} - \frac{\partial\overline{\overline{\mathbf{W}_2}}}{\partial x_1},
\end{aligned} \tag{2.11}$$

where $\overline{\overline{\mathbf{M}}} = \int_0^1 \mathbf{M}\,d\theta$, $\overline{\overline{\mathcal{H}}} = \int_0^1 \mathcal{H}\,d\theta$ *and* $\overline{\overline{\mathbf{W}}} = \int_0^1 \mathbf{W}\,d\theta$, *and equipped with initial conditions*

$$(\mathcal{I} - \Delta\mathcal{I})_{|t=0} = \tilde{\mathcal{I}}_0 + \frac{\partial(\tilde{\mathbf{N}}_0)_1}{\partial x_1} - \frac{\partial(\tilde{\mathbf{N}}_0)_2}{\partial x_2}. \tag{2.12}$$

Remark 2.1. As we shall see in the proof of this Theorem, $(\mathcal{I}, \mathbf{N})$ is also the 1-periodic two scale limit, that does not depend on the oscillating variable, of $(\tilde{\mathcal{I}}, \tilde{\mathbf{N}})$.

3. Models

In this section, we first consider a reference model. It consists in removing the ocean level and the ocean velocity which are induced by the tide wave from the Shallow Water Equations. This gives rise to a system of equations governing the time evolution of the ocean level perturbation and of the ocean velocity perturbation. Then, we analyze the scale of the variables and fields involved in the problem we want to describe. Rescaling the reference model in view of this scale analysis finally yields the desired models.

3.1. Reference model

It is generally admitted that the evolution of the ocean level $h \equiv h(t, \mathbf{x})$ (see Fig. 1) and of the ocean velocity $\mathbf{m} \equiv \mathbf{m}(t, \mathbf{x})$ is well described by the following Shallow Water Equations

$$\frac{\partial h}{\partial t} + \nabla(h - h_b) \cdot \mathbf{m} + (h - h_b)\nabla \cdot \mathbf{m} = 0, \tag{3.1}$$

$$\frac{\partial \mathbf{m}}{\partial t} + (\nabla \mathbf{m})\mathbf{m} + f\mathbf{m}^{\perp} + g\nabla h - c\Delta \mathbf{m} - c\frac{(\nabla \mathbf{m})\nabla(h - h_b)}{h - h_b} + \frac{\frac{\kappa}{h-h_b}}{1 + \frac{\kappa}{c}(h - h_b)}\mathbf{m}$$
$$= \frac{\frac{\mu}{h-h_b}}{1 + \frac{\mu}{c}(h - h_b)}(\tilde{\mathbf{W}} - \mathbf{m}) + \mathbf{F}, \tag{3.2}$$

equipped with ad-hoc initial and boundary conditions. This system was introduced by Saint-Venant [29]. For an exhaustive explanation concerning ocean modeling and the construction of this model we refer for instance to Pedlosky [28], Nihoul [27], Lions, Temam and Wang [22], Stoker [36], Whitham [39] or Johnson [16]. For a deduction of the Shallow Water Model taking into account viscosity, being able to model the consequences of wind and bottom friction actions, which is considered here, we refer to Gerbeau and Perthame [15]. In system (3.1)–(3.2), $h_b \equiv h_b(\mathbf{x})$ is the depth of the ocean bottom, f is the Coriolis parameter, g is the gravity acceleration and c is the water viscosity. The friction coefficient on the bottom is κ and the air–water friction coefficient is μ. Lastly, $\tilde{\mathbf{W}} \equiv \tilde{\mathbf{W}}(t, \mathbf{x})$ is the wind velocity and $\mathbf{F}$ may take into account the action of other meteorological factors like atmospheric pressure.

Now we isolate the action of the tide wave. In other words, we consider that the ocean depth variation $\tilde{\mathcal{H}} \equiv \tilde{\mathcal{H}}(t, \mathbf{x})$ around the mean water height $E \equiv E(\mathbf{x})$ and the ocean velocity $\tilde{\mathbf{M}} \equiv \tilde{\mathbf{M}}(t, \mathbf{x})$ which are induced by the tide wave are known. We consider that $(E + \tilde{\mathcal{H}}, \tilde{\mathbf{M}})$ is the solution to

$$\frac{\partial \tilde{\mathcal{H}}}{\partial t} + \nabla(E + \tilde{\mathcal{H}}) \cdot \tilde{\mathbf{M}} + (E + \tilde{\mathcal{H}})\nabla \cdot \tilde{\mathbf{M}} = 0, \tag{3.3}$$

$$\frac{\partial \tilde{\mathbf{M}}}{\partial t} + (\nabla \tilde{\mathbf{M}})\tilde{\mathbf{M}} + f\tilde{\mathbf{M}}^{\perp} + g\nabla(E + \tilde{\mathcal{H}} + h_b) = 0, \tag{3.4}$$

with initial and boundary conditions imposed by the tide wave.

A brief parameter size analysis, that will be confirmed in the next sections, shows that the terms $-c\Delta \mathbf{m} - c\frac{(\nabla \mathbf{m})\nabla(h-h_b)}{h-h_b} + \frac{\frac{\kappa}{h-h_b}}{1+\frac{\kappa}{c}(h-h_b)}\mathbf{m}$ have a very small influence on the sea movement. Hence we have chosen to put them in the equations for the perturbations hereafter.

Now we introduce the perturbations $\tilde{\mathcal{I}}$ and $\tilde{\mathbf{N}}$ which are defined such that $h = h_b + E + \tilde{\mathcal{H}} + \tilde{\mathcal{I}}$ and $\mathbf{m} = \tilde{\mathbf{M}} + \tilde{\mathbf{N}}$ (see Fig. 1).

Replacing h and $\mathbf{m}$ by these expressions in (3.1)–(3.2) and removing the terms appearing in (3.3)–(3.4) leads to the equations for $\tilde{\mathcal{I}}$ and $\tilde{\mathbf{N}}$. Then we obtain the following reference system which is the starting point of our scale analysis.

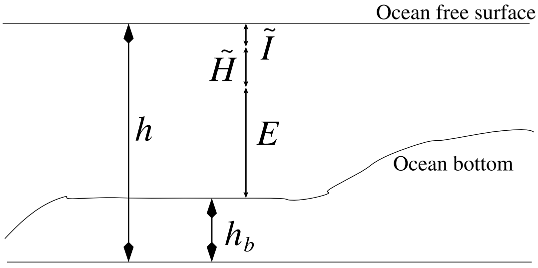

Fig. 1. Fields $h, h_b, E, \tilde{\mathcal{H}}$ and $\tilde{\mathcal{I}}$.

$$\frac{\partial \tilde{\mathcal{I}}}{\partial t} + \nabla(E+\tilde{\mathcal{H}})\cdot\tilde{\mathbf{N}} + (E+\tilde{\mathcal{H}})(\nabla\cdot\tilde{\mathbf{N}}) + (\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{M}} + \tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{M}})$$
$$+ (\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{N}} + \tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{N}}) = 0, \tag{3.5}$$

$$\frac{\partial \tilde{\mathbf{N}}}{\partial t} + (\nabla\tilde{\mathbf{N}})\tilde{\mathbf{M}} + (\nabla\tilde{\mathbf{M}})\tilde{\mathbf{N}} + (\nabla\tilde{\mathbf{N}})\tilde{\mathbf{N}} + f\tilde{\mathbf{N}}^{\perp} + g\nabla\tilde{\mathcal{I}} - c\Delta\tilde{\mathbf{M}} - c\Delta\tilde{\mathbf{N}}$$
$$- c\frac{(\nabla\tilde{\mathbf{M}})\nabla(E+\tilde{\mathcal{H}})}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}} - c\frac{(\nabla\tilde{\mathbf{M}})\nabla\tilde{\mathcal{I}}}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}} - c\frac{(\nabla\tilde{\mathbf{N}})\nabla(E+\tilde{\mathcal{H}})}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}} - c\frac{(\nabla\tilde{\mathbf{N}})\nabla\tilde{\mathcal{I}}}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}}$$
$$+ \frac{\frac{\kappa}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}}}{1+\frac{\kappa}{c}(E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}})}\tilde{\mathbf{M}} + \frac{\frac{\kappa}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}}}{1+\frac{\kappa}{c}(E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}})}\tilde{\mathbf{N}}$$
$$= \frac{\frac{\mu}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}}}{1+\frac{\mu}{c}(E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}})}(\tilde{\mathbf{W}}-\tilde{\mathbf{M}}) - \frac{\frac{\mu}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}}}{1+\frac{\mu}{c}(E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}})}\tilde{\mathbf{N}} + \mathbf{F}. \tag{3.6}$$

3.2. Rescaled variables and fields

We introduce a reference time $\bar{t}$, two reference lengths $\bar{L}$ and $\bar{l}$. Those reference values, as well as the other ones introduced hereafter, will represent characteristic values (mean or maximum values for example) of the physical quantities under consideration. We consider the rescaled variables t' and $\mathbf{x}' = (\mathbf{x}'_1, \mathbf{x}'_2)$ expressing time and position in unit $\bar{t}$, $\bar{L}$ and $\bar{l}$. They are defined as

$$t = \bar{t}t', \quad \mathbf{x}_1 = \bar{L}\mathbf{x}'_1 \quad \text{and} \quad \mathbf{x}_2 = \bar{l}\mathbf{x}'_2. \tag{3.7}$$

If the reference values are chosen as evoked above, the order of magnitude of the rescaled variables are 1. Then we define $\overline{M}$ and $\overline{N}$ as the characteristic velocity of the tide wave and its perturbation; $\overline{E}$ the characteristic value of the mean water depth, $\overline{H}$ the characteristic tidal range and $\overline{I}$ the characteristic value of its perturbation. $\overline{W}$ is the characteristic wind velocity and $\overline{F}$ the characteristic scale of the field $\mathbf{F}$. The rescaled fields have the following definitions:

$$\tilde{\mathbf{M}}'(t',\mathbf{x}') = \frac{1}{\overline{M}}\tilde{\mathbf{M}}(\bar{t}t', \bar{L}\mathbf{x}'_1, \bar{l}\mathbf{x}'_2), \qquad \tilde{\mathbf{N}}'(t',\mathbf{x}') = \frac{1}{\overline{N}}\tilde{\mathbf{N}}(\bar{t}t', \bar{L}\mathbf{x}'_1, \bar{l}\mathbf{x}'_2), \tag{3.8}$$

$$E'(\mathbf{x}') = \frac{1}{\overline{E}}E(\bar{L}\mathbf{x}'_1, \bar{l}\mathbf{x}'_2), \tag{3.9}$$

$$\tilde{\mathcal{H}}'(t',\mathbf{x}') = \frac{1}{\overline{H}}\tilde{\mathcal{H}}(\bar{t}t', \bar{L}\mathbf{x}'_1, \bar{l}\mathbf{x}'_2), \qquad \tilde{\mathcal{I}}'(t',\mathbf{x}') = \frac{1}{\overline{I}}\tilde{\mathcal{I}}(\bar{t}t', \bar{L}\mathbf{x}'_1, \bar{l}\mathbf{x}'_2), \tag{3.10}$$

$$\tilde{\mathbf{W}}'(t',\mathbf{x}') = \frac{1}{\overline{W}}\mathbf{W}(\bar{t}t', \bar{L}\mathbf{x}'_1, \bar{l}\mathbf{x}'_2), \qquad \mathbf{F}'(t',\mathbf{x}') = \frac{1}{\overline{F}}\mathbf{F}(\bar{t}t', \bar{L}\mathbf{x}'_1, \bar{l}\mathbf{x}'_2). \tag{3.11}$$

Lastly, we introduce $\overline{\omega}$ the tide wave frequency and we assume that $\tilde{\mathbf{M}}$ and $\tilde{\mathcal{H}}$ are $1/\overline{\omega}$-periodic function with modulated amplitude. In other words, we assume

$$\tilde{\mathbf{M}}'(t',\mathbf{x}') = \mathbf{M}'(t',\overline{\omega}\bar{t}t',\mathbf{x}'), \qquad \tilde{\mathcal{H}}'(t',\mathbf{x}') = \mathcal{H}'(t',\overline{\omega}\bar{t}t',\mathbf{x}'), \tag{3.12}$$

where $\theta \mapsto (\mathbf{M}'(t',\theta,\mathbf{x}'), \mathcal{H}'(t',\theta,\mathbf{x}'))$ is 1-periodic.

From system (3.5)–(3.6), we deduce the following rescaled equations for $\tilde{\mathbf{N}}'$, and $\mathcal{I}'$ with known fields $\mathbf{M}'$, E', $\mathbf{W}'$ and $\mathbf{F}'$:

$$\begin{aligned}
&\frac{\partial \tilde{\mathcal{I}}'}{\partial t'} + \frac{\overline{H}}{\bar{I}}\frac{\overline{N}\bar{t}}{\bar{L}}\left[\begin{pmatrix} \frac{\overline{E}}{\overline{H}}\frac{\partial E'}{\partial x_1'} + \frac{\partial \tilde{\mathcal{H}}'}{\partial x_1'} \\ \frac{\bar{L}}{\bar{l}}\left(\frac{\overline{E}}{\overline{H}}\frac{\partial E'}{\partial x_2'} + \frac{\partial \tilde{\mathcal{H}}'}{\partial x_2'}\right)\end{pmatrix}\cdot \tilde{\mathbf{N}}' + \left(\frac{\overline{E}}{\overline{H}}E' + \tilde{\mathcal{H}}'\right)\left(\frac{\partial \tilde{\mathbf{N}}_1'}{\partial x_1'} + \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathbf{N}}_2'}{\partial x_2'}\right)\right] + \frac{\overline{M}\bar{t}}{\bar{L}}\left[\begin{pmatrix} \frac{\partial \tilde{\mathcal{I}}'}{\partial x_1'} \\ \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathcal{I}}'}{\partial x_2'}\end{pmatrix}\cdot \tilde{\mathbf{M}}'\right. \\
&\quad \left. + \tilde{\mathcal{I}}'\left(\frac{\partial \tilde{\mathbf{M}}_1'}{\partial x_1'} + \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathbf{M}}_2'}{\partial x_2'}\right)\right] + \frac{\overline{N}\bar{t}}{\bar{L}}\left[\begin{pmatrix} \frac{\partial \tilde{\mathcal{I}}'}{\partial x_1'} \\ \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathcal{I}}'}{\partial x_2'}\end{pmatrix}\cdot \tilde{\mathbf{N}}' + \tilde{\mathcal{I}}'\left(\frac{\partial \tilde{\mathbf{N}}_1'}{\partial x_1'} + \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathbf{N}}_2'}{\partial x_2'}\right)\right] = 0,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&\frac{\partial \tilde{\mathbf{N}}'}{\partial t'} + \frac{\overline{M}\bar{t}}{\bar{L}}\left[\left(\frac{\partial \tilde{\mathbf{N}}'}{\partial x_1'}, \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathbf{N}}'}{\partial x_2'}\right)\tilde{\mathbf{M}}' + \left(\frac{\partial \tilde{\mathbf{M}}'}{\partial x_1'}, \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathbf{M}}'}{\partial x_2'}\right)\tilde{\mathbf{N}}'\right] + \frac{\overline{N}\bar{t}}{\bar{L}}\left(\frac{\partial \tilde{\mathbf{N}}'}{\partial x_1'}, \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathbf{N}}'}{\partial x_2'}\right)\tilde{\mathbf{N}}' + f\bar{t}\tilde{\mathbf{N}}'^{\perp} \\
&\quad + \frac{g\bar{t}}{\overline{N}}\frac{\bar{I}}{\bar{L}}\begin{pmatrix} \frac{\partial \tilde{\mathcal{I}}'}{\partial x_1'} \\ \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathcal{I}}'}{\partial x_2'}\end{pmatrix} - \frac{c\bar{t}}{\bar{L}^2}\frac{\overline{M}}{\overline{N}}\left(\frac{\partial^2 \tilde{\mathbf{M}}'}{\partial {x_1'}^2} + \frac{\bar{L}^2}{\bar{l}^2}\frac{\partial^2 \tilde{\mathbf{M}}'}{\partial {x_2'}^2}\right) - \frac{c\bar{t}}{\bar{L}^2}\left(\frac{\partial^2 \tilde{\mathbf{N}}'}{\partial {x_1'}^2} + \frac{\bar{L}^2}{\bar{l}^2}\frac{\partial^2 \tilde{\mathbf{N}}'}{\partial {x_2'}^2}\right) \\
&\quad - \frac{c\bar{t}}{\bar{L}^2}\frac{\overline{M}}{\overline{N}}\frac{\left(\frac{\partial \tilde{\mathbf{M}}'}{\partial x_1'}, \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathbf{M}}'}{\partial x_2'}\right)\begin{pmatrix} \frac{\partial E'}{\partial x_1'} + \frac{\overline{H}}{\overline{E}}\frac{\partial \tilde{\mathcal{H}}'}{\partial x_1'} \\ \frac{\bar{L}}{\bar{l}}\left(\frac{\partial E'}{\partial x_2'} + \frac{\overline{H}}{\overline{E}}\frac{\partial \tilde{\mathcal{H}}'}{\partial x_2'}\right)\end{pmatrix}}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'} - \frac{c\bar{t}}{\bar{L}^2}\frac{\overline{M}}{\overline{N}}\frac{\bar{I}}{\overline{E}}\frac{\left(\frac{\partial \tilde{\mathbf{M}}'}{\partial x_1'}, \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathbf{M}}'}{\partial x_2'}\right)\begin{pmatrix} \frac{\partial \tilde{\mathcal{I}}'}{\partial x_1'} \\ \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathcal{I}}'}{\partial x_2'}\end{pmatrix}}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'} \\
&\quad - \frac{c\bar{t}}{\bar{L}^2}\frac{\left(\frac{\partial \tilde{\mathbf{N}}'}{\partial x_1'}, \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathbf{N}}'}{\partial x_2'}\right)\begin{pmatrix} \frac{\partial E'}{\partial x_1'} + \frac{\overline{H}}{\overline{E}}\frac{\partial \tilde{\mathcal{H}}'}{\partial x_1'} \\ \frac{\bar{L}}{\bar{l}}\left(\frac{\partial E'}{\partial x_2'} + \frac{\overline{H}}{\overline{E}}\frac{\partial \tilde{\mathcal{H}}'}{\partial x_2'}\right)\end{pmatrix}}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'} - \frac{c\bar{t}}{\bar{L}^2}\frac{\bar{I}}{\overline{E}}\frac{\left(\frac{\partial \tilde{\mathbf{N}}'}{\partial x_1'}, \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathbf{N}}'}{\partial x_2'}\right)\begin{pmatrix} \frac{\partial \tilde{\mathcal{I}}'}{\partial x_1'} \\ \frac{\bar{L}}{\bar{l}}\frac{\partial \tilde{\mathcal{I}}'}{\partial x_2'}\end{pmatrix}}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'} \\
&\quad + \frac{\kappa\bar{t}}{\overline{E}}\frac{\overline{M}}{\overline{N}}\frac{\overline{\frac{1}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'}}}{1 + \frac{\kappa\overline{E}}{c}\left(E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'\right)}\tilde{\mathbf{M}}' + \frac{\kappa\bar{t}}{\overline{E}}\frac{\overline{\frac{1}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'}}}{1 + \frac{\kappa\overline{E}}{c}\left(E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'\right)}\tilde{\mathbf{N}}' \\
&= \frac{\mu\bar{t}}{\overline{E}}\frac{\overline{\frac{1}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'}}}{1 + \frac{\mu\overline{E}}{c}\left(E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'\right)}\left(\frac{\overline{W}}{\overline{N}}\tilde{\mathbf{W}}' - \frac{\overline{M}}{\overline{N}}\tilde{\mathbf{M}}'\right) \\
&\quad - \frac{\mu\bar{t}}{\overline{E}}\frac{\overline{\frac{1}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'}}}{1 + \frac{\mu\overline{E}}{c}\left(E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\bar{I}}{\overline{E}}\tilde{\mathcal{I}}'\right)}\tilde{\mathbf{N}}' + \frac{\overline{F}\bar{t}}{\overline{N}}\mathbf{F}'.
\end{aligned} \tag{3.14}$$

3.3. Parameter size and rescaled equations

In this subsection, we fix the characteristic values. As set out in the previous subsection, we shall choose mean values or maximal values of the concerned physical quantities. The parameter $\bar{t}$ is the observation time scale. We consider that it is about several months. Then we set

$$\bar{t} \sim 100 \text{ days} \sim 2400 \text{ h}, \tag{3.15}$$

beside this $\overline{\omega}$ is the tide frequency, meaning $1/\overline{\omega}$ is the tide duration, i.e.:

$$\frac{1}{\overline{\omega}} \sim 13 \text{ h}. \tag{3.16}$$

Hence we exhibit a small parameter:

$$\varepsilon = \frac{1}{\bar{t}\overline{\omega}} \sim \frac{1}{200}. \tag{3.17}$$

Then, we make a strong assumption, which is that $\tilde{\mathbf{N}}$ and $\tilde{\mathcal{I}}$ are really perturbations. In other words, we consider that

$$\frac{\overline{N}}{\overline{M}} \sim \frac{\bar{I}}{\overline{H}} \sim \varepsilon. \tag{3.18}$$

Concerning the Coriolis parameter, for latitudes about 45°, $f \sim \pi/\text{day} \sim 4 \times 10^{-5}/\text{s}$, then $f\bar{t} \sim \pi/2\varepsilon$. Concerning the other parameters of physical meaning, several choices are possible, according to the turbulence action, the nature of the ocean bottom, the shape of the ocean free surface and so on. We focus on one of those choices, being aware that others, that would lead to other models, are also reasonable. For the viscosity, we chose the value of the water viscosity at 20 °C, i.e. $c \sim 10^{-2}\ \text{cm}^2/\text{s} \sim 10^{-7}\ \text{km}^2/\text{day}$, then $c\bar{t} \sim 10^{-5}\ \text{km}^2$. Concerning the friction coefficients, the bottom friction coefficient is $\kappa \sim 10^{-4}\ \text{m/s} \sim 10^{-2}\ \text{km/day}$ and the air–water friction coefficient is $\mu \sim 10^{-6}\ \text{m/s} \sim 10^{-4}\ \text{km/day}$. Those values are consistent with the ones used, for instance, in Dawson and Proft [8] or Gerbeau and Perthame [15]. Then $\kappa\bar{t} \sim 1$ km, $\mu\bar{t} \sim 10^{-2}$ km, $\kappa/c \sim 10^5/\text{km}$ and $\mu/c \sim 10^3/\text{km}$. We also have $g \sim 10\ \text{m/s}^2 \sim 10^6\ \text{km/day}^2$ and $g\bar{t} \sim 10^8$ km/day.

We now turn to the ratios determining the asymptotic analysis we have to realize. Having in mind our final target, i.e. the drift of things in the ocean over long time periods, we notice that such a drift may take place relatively far from the coast, above the **continental shelf**. It may also take place in a large and relatively closed bay, with a long residence time of the ocean water. Such a domain will be called **coastal zone**. As was the case in 1999/2000 for the Erika oil slick along the French Atlantic coast, the drift may occur for weeks along a thin **layer** following the coast. Those remarks guide the choices concerning the geometrical assumptions we consider further.

$\bar{L}$ and $\bar{l}$ represent the characteristic lengths of the domain where the drift takes place and $\overline{M}/\overline{\omega}$ the characteristic distance the water covers in the tide duration. Following Salomon and Breton [30], Četina, Rajar and Povinec [38], Bao, Gao and Yan [5] or Cai, Huang and Long [7], we can state that this distance is about a few kilometers (from 5 to 20) in the cases we are interested in. If the domain under consideration is a **continental shelf**, the characteristic sea water velocity $\overline{M}$ is about 0.5 km/h and then, we have

$$\frac{\overline{M}}{\overline{\omega}} \sim 5 \text{ km}, \tag{3.19}$$

and if we set $\bar{L} \sim \bar{l} \sim 500$ km, $\bar{E} \sim 300$ m and $\overline{H} \sim 3$ m, then

$$\frac{\frac{\overline{M}}{\overline{\omega}}}{\bar{L}} \sim 2\varepsilon, \qquad \frac{\overline{H}}{\bar{E}} \sim 2\varepsilon. \tag{3.20}$$

We also have $\overline{N} \sim \varepsilon\overline{M} \sim 2.5 \times 10^{-3}\ \text{km/h} \sim 6 \times 10^{-2}\ \text{km/day}$, then we get $g\bar{t}/\overline{N} \sim 1.7 \times 10^9$. Since $\bar{I} \sim \varepsilon\overline{H} \sim 1.5 \times 10^{-2}$ m, we obtain $\bar{I}/\bar{L} \sim 3 \times 10^{-8}$. Hence

$$\frac{g\bar{t}}{\overline{N}}\frac{\bar{I}}{\bar{L}} \sim 50 \sim \frac{1}{4\varepsilon}. \tag{3.21}$$

Moreover

$$\frac{c\bar{t}}{\bar{L}^2} \sim \frac{10^{-5}}{25\times 10^4} \sim 13\varepsilon^5, \tag{3.22}$$

$$\frac{\kappa\bar{t}}{\overline{E}} \sim \frac{1}{0.3} \sim 3.3 \sim 3, \qquad \frac{\kappa\overline{E}}{c} \sim 3\times 10^4 \sim \frac{0.8}{\varepsilon^2}, \tag{3.23}$$

$$\frac{\mu\bar{t}}{\overline{E}} \sim \frac{10^{-2}}{0.3} \sim 3.3\times 10^{-2} \sim 6\varepsilon, \qquad \frac{\mu\overline{E}}{c} \sim 3\times 10^2 \sim \frac{1.5}{\varepsilon}. \tag{3.24}$$

Concerning the wind velocity, when the weather is calm, 10 km/h is a relevant characteristic value, while 100 km/h may be a good choice in stormy conditions. Hence, we shall consider

$$\frac{\overline{M}}{\overline{W}} \sim \frac{0.5}{10} \sim \frac{1}{20}, \tag{3.25}$$

in calm weather regime, and,

$$\frac{\overline{M}}{\overline{W}} \sim \frac{0.5}{100} \sim \varepsilon, \tag{3.26}$$

in storm regime. Expressing now the following ratios

$$\frac{\overline{M}\bar{t}}{\bar{L}} \sim \bar{t}\overline{\omega}\frac{\frac{\overline{M}}{\overline{\omega}}}{\bar{L}}, \qquad \frac{\overline{N}\bar{t}}{\bar{L}} \sim \frac{\overline{M}\bar{t}}{\bar{L}}\frac{\overline{N}}{\overline{M}}, \tag{3.27}$$

and moreover setting $\overline{F}\bar{t} \sim \overline{N}$ and removing the $'$ for clarity, we can write the rescaled Eqs. (3.13)–(3.14) in the case of a continental shelf:

$$\begin{aligned}
&\frac{\partial\tilde{\mathcal{I}}}{\partial t} + \nabla\left(\frac{1}{\varepsilon}E + 2\tilde{\mathcal{H}}\right)\cdot\tilde{\mathbf{N}} + \left(\frac{1}{\varepsilon}E + 2\tilde{\mathcal{H}}\right)\nabla\cdot\tilde{\mathbf{N}} + 2(\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{M}} + 2\tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{M}}) \\
&\quad + 2\varepsilon\big((\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{N}} + \tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{N}})\big) = 0,
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
&\frac{\partial\tilde{\mathbf{N}}}{\partial t} + 2(\nabla\tilde{\mathbf{N}})\tilde{\mathbf{M}} + 2(\nabla\tilde{\mathbf{M}})\tilde{\mathbf{N}} + 2\varepsilon(\nabla\tilde{\mathbf{N}})\tilde{\mathbf{N}} + \frac{\pi}{2\varepsilon}\tilde{\mathbf{N}}^{\perp} + \frac{1}{4\varepsilon}\nabla\tilde{\mathcal{I}} - 13\varepsilon^4\Delta\tilde{\mathbf{M}} - 13\varepsilon^5\Delta\tilde{\mathbf{N}} \\
&\quad - 13\varepsilon^4\frac{(\nabla\tilde{\mathbf{M}})\nabla(E+2\varepsilon\tilde{\mathcal{H}})}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}} - 26\varepsilon^6\frac{(\nabla\tilde{\mathbf{M}})\nabla\tilde{\mathcal{I}}}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}} - 13\varepsilon^5\frac{(\nabla\tilde{\mathbf{N}})\nabla(E+2\varepsilon\tilde{\mathcal{H}})}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}} \\
&\quad - 26\varepsilon^7\frac{(\nabla\tilde{\mathbf{N}})\nabla\tilde{\mathcal{I}}}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}} \\
&\quad + \frac{3}{\varepsilon}\frac{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}{1+\frac{0.8}{\varepsilon^2}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}\tilde{\mathbf{M}} + 3\frac{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}{1+\frac{0.8}{\varepsilon^2}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}\tilde{\mathbf{N}} \\
&= 6\frac{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}{1+\frac{1.5}{\varepsilon}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}(\gamma\tilde{\mathbf{W}}-\tilde{\mathbf{M}}) - 6\varepsilon\frac{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}{1+\frac{1.5}{\varepsilon}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}\tilde{\mathbf{N}} + \mathbf{F},
\end{aligned} \tag{3.29}$$

where $\gamma = 20 = 1/(10\varepsilon)$ in calm weather regime. In storm regime, which is what it is supposed for model (2.3)–(2.4) presented in the introduction, $\gamma = 1/\varepsilon$.

If the domain is a **coastal zone**, $\overline{M} \sim 1$ km/h

$$\frac{\overline{M}}{\overline{\omega}} \sim 10\ \text{km}, \tag{3.30}$$

and we set $\bar{L} \sim \bar{l} \sim 5$ km, $\bar{E} \sim 50$ m and $\overline{H} \sim 10$ m. In this case

$$\frac{\frac{\overline{M}}{\overline{\omega}}}{\bar{L}} \sim 2, \qquad \frac{\overline{H}}{\bar{E}} \sim \frac{1}{5}. \tag{3.31}$$

We also have $\overline{N} \sim \varepsilon\overline{M} \sim 1.2 \times 10^{-1}$ km/day, then we get $g\bar{t}/\overline{N} \sim 8 \times 10^8$. Since $\bar{I} \sim \varepsilon\overline{H} \sim 5 \times 10^{-2}$ m we obtain $\bar{I}/\bar{L} \sim 10^{-5}$. Hence

$$\frac{g\bar{t}}{\overline{N}}\frac{\bar{I}}{\bar{L}} \sim 8 \times 10^3 \sim \frac{0.2}{\varepsilon^2}. \tag{3.32}$$

Moreover

$$\frac{c\bar{t}}{\bar{L}^2} \sim \frac{10^{-5}}{25} \sim 0.6\varepsilon^3, \tag{3.33}$$

$$\frac{\kappa\bar{t}}{\bar{E}} \sim \frac{1}{0.05} \sim 20 \sim \frac{1}{10\varepsilon}, \qquad \frac{\kappa\bar{E}}{c} \sim 5 \times 10^3 \sim \frac{1}{10\varepsilon^2}, \tag{3.34}$$

$$\frac{\mu\bar{t}}{\bar{E}} \sim \frac{10^{-2}}{0.05} \sim 0.2, \qquad \frac{\mu\bar{E}}{c} \sim 50 \sim \frac{1}{4\varepsilon}. \tag{3.35}$$

Concerning the wind velocity, we have

$$\frac{\overline{M}}{\overline{W}} \sim \frac{1}{10}, \tag{3.36}$$

in calm weather regime, and, in storm regime

$$\frac{\overline{M}}{\overline{W}} \sim \frac{1}{100} \sim 2\varepsilon. \tag{3.37}$$

Hence, the rescaled equation reads in this case:

$$\begin{aligned}
&\frac{\partial\tilde{\mathcal{I}}}{\partial t} + \frac{2}{\varepsilon}\big(\nabla(5E+\tilde{\mathcal{H}})\big)\cdot\tilde{\mathbf{N}} + \frac{2}{\varepsilon}(5E+\tilde{\mathcal{H}})\nabla\cdot\tilde{\mathbf{N}} + \frac{2}{\varepsilon}(\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{M}} + \frac{2}{\varepsilon}\tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{M}}) \\
&\quad + 2(\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{N}} + 2\tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{N}}) = 0, \qquad (3.38)\\
&\frac{\partial\tilde{\mathbf{N}}}{\partial t} + \frac{2}{\varepsilon}(\nabla\tilde{\mathbf{N}})\tilde{\mathbf{M}} + \frac{2}{\varepsilon}(\nabla\tilde{\mathbf{M}})\tilde{\mathbf{N}} + 2(\nabla\tilde{\mathbf{N}})\tilde{\mathbf{N}} + \frac{\pi}{2\varepsilon}\tilde{\mathbf{N}}^{\perp} + \frac{0.2}{\varepsilon^2}\nabla\tilde{\mathcal{I}} - 0.6\varepsilon^2\Delta\tilde{\mathbf{M}} - 0.6\varepsilon^3\Delta\tilde{\mathbf{N}} \\
&\quad - 0.6\varepsilon^2\frac{(\nabla\tilde{\mathbf{M}})\nabla(E+\tilde{\mathcal{H}})}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}} - 0.1\varepsilon^2\frac{(\nabla\tilde{\mathbf{M}})\nabla\tilde{\mathcal{I}}}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}'} - 0.6\varepsilon^2\frac{(\nabla\tilde{\mathbf{N}})\nabla(E+\tilde{\mathcal{H}})}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}}
\end{aligned}$$

$$- 0.1\varepsilon^3 \frac{(\nabla\tilde{\mathbf{N}})\nabla\tilde{\mathcal{I}}}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}} + \frac{1}{10}\frac{\frac{1}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}}}{1+\frac{1}{10\varepsilon^2}\left(E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}\right)}\tilde{\mathbf{M}} + \frac{1}{10\varepsilon}\frac{\frac{1}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}}}{1+\frac{1}{10\varepsilon^2}\left(E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}\right)}\tilde{\mathbf{N}}$$

$$= 0.2\frac{\frac{1}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}}}{1+\frac{1}{4\varepsilon}\left(E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}\right)}\left(\frac{\gamma}{2\varepsilon}\tilde{\mathbf{W}} - \frac{1}{\varepsilon}\tilde{\mathbf{M}}\right) - 0.2\frac{\frac{1}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}}}{1+\frac{1}{\varepsilon}\left(E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\varepsilon\tilde{\mathcal{I}}\right)}\tilde{\mathbf{N}} + \mathbf{F}, \tag{3.39}$$

where $\gamma/2 = 10 = 1/20\varepsilon$ in still weather and $\gamma/2 = 1/2\varepsilon$ in stormy weather.

We will give the name of **coastal layer** to a domain having the following characteristics

$$\frac{\overline{M}}{\overline{\omega}} \sim 10\ \text{km}, \tag{3.40}$$

and $\bar{L} \sim 500$ km, $\bar{l} \sim 5$ km, $\overline{E} \sim 50$ m and $\overline{H} \sim 10$ m and then

$$\frac{\frac{\overline{M}}{\overline{\omega}}}{\bar{L}} \sim 4\varepsilon, \qquad \frac{\bar{l}}{\bar{L}} \sim 2\varepsilon, \qquad \frac{\overline{H}}{\overline{E}} \sim \frac{1}{5}. \tag{3.41}$$

In this case, we have $\overline{N} \sim \varepsilon\overline{M} \sim 1.2 \times 10^{-1}$ km/day, then we get $g\bar{t}/\overline{N} \sim 8 \times 10^8$. Since $\bar{I} \sim \varepsilon\overline{H} \sim 5 \times 10^{-2}$ m, we obtain $\bar{I}/\bar{L} \sim 10^{-7}$. Hence

$$\frac{g\bar{t}}{\overline{N}}\frac{\bar{I}}{\bar{L}} \sim 80 \sim \frac{0.4}{\varepsilon}. \tag{3.42}$$

Moreover

$$\frac{c\bar{t}}{\bar{L}^2} \sim \frac{10^{-5}}{25 \times 10^4} \sim 13\varepsilon^5, \tag{3.43}$$

$$\frac{\kappa\bar{t}}{\overline{E}} \sim \frac{1}{0.05} \sim 20 \sim \frac{1}{10\varepsilon}, \qquad \frac{\kappa\overline{E}}{c} \sim 5 \times 10^3 \sim \frac{1}{10\varepsilon^2}, \tag{3.44}$$

$$\frac{\mu\bar{t}}{\overline{E}} \sim \frac{10^{-2}}{0.05} \sim 0.2, \qquad \frac{\mu\overline{E}}{c} \sim 50 \sim \frac{1}{4\varepsilon}. \tag{3.45}$$

The considerations concerning the wind velocity are the same as in the case of a coastal zone. The rescaled equation for a coastal layer writes:

$$\begin{aligned}
&\frac{\partial\tilde{\mathcal{I}}}{\partial t} + \begin{pmatrix} 20\frac{\partial E}{\partial x_1} + \frac{\partial\tilde{\mathcal{H}}}{\partial x_1} \\ \frac{2}{\varepsilon}\left(5\frac{\partial E}{\partial x_2} + \frac{\partial\tilde{\mathcal{H}}}{\partial x_2}\right)\end{pmatrix}\cdot\tilde{\mathbf{N}} + (20E+\tilde{\mathcal{H}})\left(\frac{\partial\tilde{\mathbf{N}}_1}{\partial x_1} + \frac{1}{2\varepsilon}\frac{\partial\tilde{\mathbf{N}}_2}{\partial x_2}\right) + 4\begin{pmatrix}\frac{\partial\tilde{\mathcal{I}}}{\partial x_1} \\ \frac{1}{2\varepsilon}\frac{\partial\tilde{\mathcal{I}}}{\partial x_2}\end{pmatrix}\cdot\tilde{\mathbf{M}} \\
&\quad + 4\tilde{\mathcal{I}}\left(\frac{\partial\tilde{\mathbf{M}}_1}{\partial x_1} + \frac{1}{2\varepsilon}\frac{\partial\tilde{\mathbf{M}}_2}{\partial x_2}\right) + \begin{pmatrix}4\varepsilon\frac{\partial\tilde{\mathcal{I}}}{\partial x_1} \\ 2\frac{\partial\tilde{\mathcal{I}}}{\partial x_2}\end{pmatrix}\cdot\tilde{\mathbf{N}} + \tilde{\mathcal{I}}\left(4\varepsilon\frac{\partial\tilde{\mathbf{N}}_1}{\partial x_1} + 2\frac{\partial\tilde{\mathbf{N}}_2}{\partial x_2}\right) = 0,
\end{aligned} \tag{3.46}$$

$$\begin{aligned}
&\frac{\partial\tilde{\mathbf{N}}}{\partial t} + \left(4\frac{\partial\tilde{\mathbf{N}}}{\partial x_1}, \frac{2}{\varepsilon}\frac{\partial\tilde{\mathbf{N}}}{\partial x_2}\right)\tilde{\mathbf{M}} + \left(4\frac{\partial\tilde{\mathbf{M}}}{\partial x_1}, \frac{2}{\varepsilon}\frac{\partial\tilde{\mathbf{M}}}{\partial x_2}\right)\tilde{\mathbf{N}} + \left(4\varepsilon\frac{\partial\tilde{\mathbf{N}}}{\partial x_1}, 2\frac{\partial\tilde{\mathbf{N}}}{\partial x_2}\right)\tilde{\mathbf{N}} + \frac{\pi}{2\varepsilon}\tilde{\mathbf{N}}^{\perp} \\
&\quad + \frac{0.4}{\varepsilon}\begin{pmatrix}\frac{\partial\tilde{\mathcal{I}}}{\partial x_1} \\ \frac{1}{2\varepsilon}\frac{\partial\tilde{\mathcal{I}}}{\partial x_2}\end{pmatrix} - \left(13\varepsilon^4\frac{\partial^2\tilde{\mathbf{M}}}{\partial x_1{}^2} + \frac{13\varepsilon^2}{4}\frac{\partial^2\tilde{\mathbf{M}}}{\partial x_2{}^2}\right) - \left(13\varepsilon^5\frac{\partial^2\tilde{\mathbf{N}}}{\partial x_1{}^2} + \frac{13\varepsilon^3}{4}\frac{\partial^2\tilde{\mathbf{N}}}{\partial x_2{}^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&-13\varepsilon^4 \frac{\left(\frac{\partial\tilde{\mathbf{M}}}{\partial x_1}, \frac{1}{2\varepsilon}\frac{\partial\tilde{\mathbf{M}}}{\partial x_2}\right)\begin{pmatrix} \frac{\partial E}{\partial x_1} + \frac{1}{5}\frac{\partial\tilde{\mathcal{H}}}{\partial x_1} \\ \frac{1}{2\varepsilon}\left(\frac{\partial E}{\partial x_2} + \frac{1}{5}\frac{\partial\tilde{\mathcal{H}}}{\partial x_2}\right)\end{pmatrix}}{E + \frac{1}{5}\tilde{\mathcal{H}} + \frac{\varepsilon}{5}\tilde{\mathcal{I}}} - \frac{13\varepsilon^5}{5}\frac{\left(\frac{\partial\tilde{\mathbf{M}}}{\partial x_1}, \frac{1}{2\varepsilon}\frac{\partial\tilde{\mathbf{M}}}{\partial x_2}\right)\begin{pmatrix} \frac{\partial\tilde{\mathcal{I}}}{\partial x_1} \\ \frac{1}{2\varepsilon}\frac{\partial\tilde{\mathcal{I}}}{\partial x_2}\end{pmatrix}}{E + \frac{1}{5}\tilde{\mathcal{H}} + \frac{\varepsilon}{5}\tilde{\mathcal{I}}} \\
&-13\varepsilon^5 \frac{\left(\frac{\partial\tilde{\mathbf{N}}}{\partial x_1}, \frac{1}{2\varepsilon}\frac{\partial\tilde{\mathbf{N}}}{\partial x_2}\right)\begin{pmatrix} \frac{\partial E}{\partial x_1} + \frac{1}{5}\frac{\partial\tilde{\mathcal{H}}}{\partial x_1} \\ \frac{1}{2\varepsilon}\left(\frac{\partial E}{\partial x_2} + \frac{1}{5}\frac{\partial\tilde{\mathcal{H}}}{\partial x_2}\right)\end{pmatrix}}{E + \frac{1}{5}\tilde{\mathcal{H}} + \frac{\varepsilon}{5}\tilde{\mathcal{I}}} - \frac{13\varepsilon^6}{5}\frac{\left(\frac{\partial\tilde{\mathbf{N}}}{\partial x_1}, \frac{1}{2\varepsilon}\frac{\partial\tilde{\mathbf{N}}}{\partial x_2}\right)\begin{pmatrix} \frac{\partial\tilde{\mathcal{I}}}{\partial x_1} \\ \frac{1}{2\varepsilon}\frac{\partial\tilde{\mathcal{I}}}{\partial x_2'}\end{pmatrix}}{E + \frac{1}{5}\tilde{\mathcal{H}} + \frac{\varepsilon}{5}\tilde{\mathcal{I}}} \\
&+\frac{1}{10\varepsilon^2}\frac{\frac{1}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}}}{1 + \frac{1}{10\varepsilon^2}\left(E + \frac{1}{5}\tilde{\mathcal{H}} + \frac{\varepsilon}{5}\tilde{\mathcal{I}}\right)}\tilde{\mathbf{M}} + \frac{1}{10\varepsilon}\frac{\frac{1}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}}}{1 + \frac{1}{10\varepsilon^2}\left(E + \frac{1}{5}\tilde{\mathcal{H}} + \frac{\varepsilon}{5}\tilde{\mathcal{I}}\right)}\tilde{\mathbf{N}} \\
&= 0.2\frac{\frac{1}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}}}{1 + \frac{1}{4\varepsilon}\left(E + \frac{1}{5}\tilde{\mathcal{H}} + \frac{\varepsilon}{5}\tilde{\mathcal{I}}\right)}\left(\frac{\gamma}{2\varepsilon}\tilde{\mathbf{W}} - \frac{1}{\varepsilon}\tilde{\mathbf{M}}\right) - 0.2\frac{\frac{1}{E+\frac{1}{5}\tilde{\mathcal{H}}+\frac{\varepsilon}{5}\tilde{\mathcal{I}}}}{1 + \frac{1}{4\varepsilon}\left(E + \frac{1}{5}\tilde{\mathcal{H}} + \frac{\varepsilon}{5}\tilde{\mathcal{I}}\right)}\tilde{\mathbf{N}} + \mathbf{F}. \qquad (3.47)
\end{aligned}
$$

4. Existence

4.1. Simplified system for continental shelf

In this section, we focus on one of the models introduced in the previous section, and we explore some of its mathematical properties.

More precisely, we consider a simplified version of system (3.28)–(3.29) which consists in considering that the ocean bottom is flat, i.e. $E \equiv 1$, in forgetting all the power of ε greater than 1 and in setting all constants to 1. Then we obtain system (2.5)–(2.6) and we prove an existence result for the solution of this system.

Although the results given in Sections 4 and 5 are specific to the model (2.5)–(2.6), we expect that similar methods could be used to prove similar results for the other models introduced in Section 3.

4.2. Proof of Theorem 2.1

Setting $\mathbf{u} = (\tilde{\mathcal{I}}, \tilde{\mathbf{N}}) = (\tilde{\mathcal{I}}, \tilde{\mathbf{N}}_1, \tilde{\mathbf{N}}_2)$, $\mathbf{u}^\perp = (0, \tilde{\mathbf{N}}^\perp)$ and introducing

$$
B^1\left(t, \frac{t}{\varepsilon}, \mathbf{x}, \varepsilon\mathbf{u}\right) = \begin{pmatrix} \tilde{\mathbf{M}}_1 + \varepsilon\tilde{\mathbf{N}}_1 & \frac{1}{\varepsilon} + \tilde{\mathcal{H}} + \varepsilon\tilde{\mathcal{I}} & 0 \\ \frac{1}{\varepsilon} & \tilde{\mathbf{M}}_1 + \varepsilon\tilde{\mathbf{N}}_1 & 0 \\ 0 & 0 & \tilde{\mathbf{M}}_1 + \varepsilon\tilde{\mathbf{N}}_1 \end{pmatrix}, \qquad (4.1)
$$

$$
B^2\left(t, \frac{t}{\varepsilon}, \mathbf{x}, \varepsilon\mathbf{u}\right) = \begin{pmatrix} \tilde{\mathbf{M}}_2 + \varepsilon\tilde{\mathbf{N}}_2 & 0 & \frac{1}{\varepsilon} + \tilde{\mathcal{H}} + \varepsilon\tilde{\mathcal{I}} \\ 0 & \tilde{\mathbf{M}}_2 + \varepsilon\tilde{\mathbf{N}}_2 & 0 \\ \frac{1}{\varepsilon} & 0 & \tilde{\mathbf{M}}_2 + \varepsilon\tilde{\mathbf{N}}_2 \end{pmatrix}, \qquad (4.2)
$$

and

$$
F\left(t, \frac{t}{\varepsilon}, \mathbf{x}, \mathbf{u}\right) = \begin{pmatrix} -\left(\frac{\partial\tilde{\mathcal{H}}}{\partial x_1}\tilde{\mathbf{N}}_1 + \frac{\partial\tilde{\mathcal{H}}}{\partial x_2}\tilde{\mathbf{N}}_2\right) - \left(\frac{\partial\tilde{\mathbf{M}}_1}{\partial x_1} + \frac{\partial\tilde{\mathbf{M}}_2}{\partial x_2}\right)\tilde{\mathcal{I}} \\ \tilde{\mathbf{W}}_1 - \left(\frac{\partial\tilde{\mathbf{M}}_1}{\partial x_1}\tilde{\mathbf{N}}_1 + \frac{\partial\tilde{\mathbf{M}}_1}{\partial x_2}\tilde{\mathbf{N}}_2\right) \\ \tilde{\mathbf{W}}_2 - \left(\frac{\partial\tilde{\mathbf{M}}_2}{\partial x_1}\tilde{\mathbf{N}}_1 + \frac{\partial\tilde{\mathbf{M}}_2}{\partial x_2}\tilde{\mathbf{N}}_2\right) \end{pmatrix}, \qquad (4.3)
$$

Eqs. (2.5)–(2.6) read

$$\frac{\partial \mathbf{u}}{\partial t}+B^1\frac{\partial \mathbf{u}}{\partial x_1}+B^2\frac{\partial \mathbf{u}}{\partial x_2}+\frac{1}{\varepsilon}\mathbf{u}^\perp=F. \tag{4.4}$$

And, introducing

$$A^0\left(t,\frac{t}{\varepsilon},\mathbf{x},\varepsilon^2\tilde{\mathcal{I}}\right)=\begin{pmatrix}\frac{1}{1+\varepsilon\tilde{\mathcal{H}}+\varepsilon^2\tilde{\mathcal{I}}} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{4.5}$$

$$A^1\left(t,\frac{t}{\varepsilon},\mathbf{x},\varepsilon\mathbf{u}\right)=\begin{pmatrix}\frac{\tilde{\mathbf{M}}_1+\varepsilon\tilde{\mathbf{N}}_1}{1+\varepsilon\tilde{\mathcal{H}}+\varepsilon^2\tilde{\mathcal{I}}} & 0 & 0\\ 0 & \tilde{\mathbf{M}}_1+\varepsilon\tilde{\mathbf{N}}_1 & 0\\ 0 & 0 & \tilde{\mathbf{M}}_1+\varepsilon\tilde{\mathbf{N}}_1\end{pmatrix}, \tag{4.6}$$

$$A^2\left(t,\frac{t}{\varepsilon},\mathbf{x},\varepsilon\mathbf{u}\right)=\begin{pmatrix}\frac{\tilde{\mathbf{M}}_2+\varepsilon\tilde{\mathbf{N}}_2}{1+\varepsilon\tilde{\mathcal{H}}+\varepsilon^2\tilde{\mathcal{I}}} & 0 & 0\\ 0 & \tilde{\mathbf{M}}_2+\varepsilon\tilde{\mathbf{N}}_2 & 0\\ 0 & 0 & \tilde{\mathbf{M}}_2+\varepsilon\tilde{\mathbf{N}}_2\end{pmatrix}, \tag{4.7}$$

$$S^1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\quad\text{and}\quad S^2=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \tag{4.8}$$

Eq. (4.4) yields the following symmetric hyperbolic system:

$$A^0\frac{\partial \mathbf{u}}{\partial t}+A^1\frac{\partial \mathbf{u}}{\partial x_1}+A^2\frac{\partial \mathbf{u}}{\partial x_2}+\frac{1}{\varepsilon}S^1\frac{\partial \mathbf{u}}{\partial x_1}+\frac{1}{\varepsilon}S^2\frac{\partial \mathbf{u}}{\partial x_2}+\frac{1}{\varepsilon}\mathbf{u}^\perp=F_0=A^0F. \tag{4.9}$$

Hence applying Kato [18] or Majda [23], we deduce that for any ε, under the assumptions of Theorem 2.1, the classical solution of (2.5), (2.6) and (2.8) exists and is unique on a time interval. what remains to prove is that this time interval does not depend on ε.

Before going further in the proof, we may observe the following differences between Eq. (4.9) and the type of problems, also depending on a small parameter, studied in Klainerman and Majda [19,20], Schochet [33,32,34], and Métivier and Schochet [24]. Some of those differences simplify the problem: the nonlinearity in A^0 are functions of only $\varepsilon^2\tilde{\mathcal{I}}$ and in A^1 and A^2 of only $\varepsilon\mathbf{u}$. Some others make the results proved by those authors unable to be applied directly: A^0, A^1 and A^2 depend on t/ε and the singular term $\mathbf{u}^\perp/\varepsilon$ involves the function $\mathbf{u}$ itself and not order 1 derivatives of it. Nonetheless, the now classical calculus procedures carried out in the concerned papers may be followed in order to obtain the right estimates allowing for the conclusion. We sketch the concerned computations hereafter.

We set $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$ with $|\alpha|=\alpha_1+\alpha_2\leqslant s$ and $D^\alpha\mathbf{u}=\frac{\partial^{|\alpha|}\mathbf{u}}{\partial {x_1}^{\alpha_1}\partial {x_2}^{\alpha_2}}$. Applying D^α to Eq. (4.9) yields

$$A^0\frac{\partial D^\alpha\mathbf{u}}{\partial t}+A^1\frac{\partial D^\alpha\mathbf{u}}{\partial x_1}+A^2\frac{\partial D^\alpha\mathbf{u}}{\partial x_2}+\frac{1}{\varepsilon}S^1\frac{\partial D^\alpha\mathbf{u}}{\partial x_1}+\frac{1}{\varepsilon}S^2\frac{\partial D^\alpha\mathbf{u}}{\partial x_2}+\frac{1}{\varepsilon}\left(D^\alpha\mathbf{u}\right)^\perp=F^\alpha, \tag{4.10}$$

with

$$F^\alpha=D^\alpha F_0-\left[D^\alpha,A^0\frac{\partial}{\partial t}\right]\mathbf{u}-\left[D^\alpha,A^1\frac{\partial}{\partial x_1}\right]\mathbf{u}-\left[D^\alpha,A^2\frac{\partial}{\partial x_2}\right]\mathbf{u}, \tag{4.11}$$

[,] standing for the classical commutator.

Multiplying Eq. (4.10) by $2D^\alpha \mathbf{u}$, integrating on $\mathbb{R}^2$ and noticing that

$$2\int A^0 \frac{\partial D^\alpha \mathbf{u}}{\partial t} \cdot D^\alpha \mathbf{u}\, d\mathbf{x} = \frac{d(\int A^0 D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u}\, d\mathbf{x})}{dt} - \int \frac{d(A^0)}{dt} D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u}\, d\mathbf{x}, \tag{4.12}$$

$$\begin{aligned} 2\int A^j \frac{\partial D^\alpha \mathbf{u}}{\partial x_j} \cdot D^\alpha \mathbf{u}\, d\mathbf{x} &= \left(\int \frac{d(A^j D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u})}{dx_j}\, d\mathbf{x}\right) - \int \frac{d(A^j)}{dx_j} D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u}\, d\mathbf{x} \\ &= -\int \frac{d(A^j)}{dx_j} D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u}\, d\mathbf{x}, \end{aligned} \tag{4.13}$$

$$2\int S^j \frac{\partial D^\alpha \mathbf{u}}{\partial x_j} \cdot D^\alpha \mathbf{u}\, d\mathbf{x} = -2\int S^j \frac{\partial D^\alpha \mathbf{u}}{\partial x_j} \cdot D^\alpha\, d\mathbf{x} = 0, \tag{4.14}$$

for $j = 1, 2$, and

$$2\int \left(D^\alpha \mathbf{u}\right)^\perp \cdot D^\alpha \mathbf{u}\, d\mathbf{x} = 0, \tag{4.15}$$

we obtain

$$\begin{aligned} \frac{d(\int A^0 D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u}\, d\mathbf{x})}{dt} &= \int \frac{d(A^0)}{dt} D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u}\, d\mathbf{x} + \int \frac{d(A^1)}{dx_1} D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u}\, d\mathbf{x} \\ &\quad + \int \frac{d(A^2)}{dx_2} D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u}\, d\mathbf{x} + \int F^\alpha \cdot D^\alpha \mathbf{u}\, d\mathbf{x}. \end{aligned} \tag{4.16}$$

For all the estimates to come, all the constants which are needed are called c. Since the dependency of A^0 with respect to t/ε is done through $\varepsilon\tilde{\mathcal{H}}$, and since $s > 3$, we can deduce that for any t and $\mathbf{x}$,

$$\left|\frac{d(A^0_{11}(t, \frac{t}{\varepsilon}, \mathbf{x}, \varepsilon^2\tilde{\mathcal{I}}))}{dt}\right| \leqslant c\left(1 + \varepsilon^2\left|\frac{\partial\tilde{\mathcal{I}}}{\partial t}\right|\right) \leqslant c\left(1 + \varepsilon^2 \sup_{\mathbf{x}\in\mathbb{R}^2}\left|\frac{\partial\tilde{\mathcal{I}}}{\partial t}\right|\right) \leqslant c\left(1 + \varepsilon^2\left\|\frac{\partial\tilde{\mathcal{I}}}{\partial t}\right\|_{s-1}\right), \tag{4.17}$$

where $\| \|_{s-1}$ stands for the norm in $H^{s-1}(\mathbb{R}^2)$. The time derivatives of the other entries of A^0 are zero. Hence the first term of the right-hand side of (4.16) may be estimated

$$\left|\int \frac{d(A^0)}{dt} D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u}\, d\mathbf{x}\right| \leqslant c\left(1 + \varepsilon^2\left\|\frac{\partial\tilde{\mathcal{I}}}{\partial t}\right\|_{s-1}\right)\left\|D^\alpha \mathbf{u}\right\|_0^2 \leqslant c\left(1 + \varepsilon^2\left\|\frac{\partial\tilde{\mathcal{I}}}{\partial t}\right\|_{s-1}\right)\|\mathbf{u}\|_s^2, \tag{4.18}$$

where $\| \|_s$ stands for the norm in $(H^s(\mathbb{R}^2))^2$ and $\| \|_0$ for the norm in $(L^2(\mathbb{R}^2))^2$. Concerning the entries A^i_{kl} of A^i for $i = 1, 2$,

$$\begin{aligned} \left|\frac{d(A^i_{kl}(t, \frac{t}{\varepsilon}, \mathbf{x}, \varepsilon\mathbf{u}))}{dx_i}\right| &\leqslant c\left(1 + \varepsilon\left|\frac{\partial\tilde{\mathbf{N}}_i}{\partial x_i}\right| + \varepsilon^2\left|\frac{\partial\tilde{\mathcal{I}}}{\partial x_i}\right|\right) \leqslant c\left(1 + \varepsilon \sup_{\mathbf{x}\in\mathbb{R}^2}\left|\frac{\partial\tilde{\mathbf{N}}_i}{\partial x_i}\right| + \varepsilon^2 \sup_{\mathbf{x}\in\mathbb{R}^2}\left|\frac{\partial\tilde{\mathcal{I}}}{\partial x_i}\right|\right) \\ &\leqslant c\left(1 + \varepsilon\left\|\frac{\partial\mathbf{u}}{\partial x_i}\right\|_{s-1}\right) \leqslant c\left(1 + \varepsilon\|\mathbf{u}\|_s\right). \end{aligned} \tag{4.19}$$

Hence

$$\left| \int \frac{d(A^i)}{dx_i} D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u} \, d\mathbf{x} \right| \leqslant c\big(1 + \varepsilon \|\mathbf{u}\|_s\big) \big\| D^\alpha \mathbf{u} \big\|_0^2 \leqslant c\big(1 + \varepsilon \|\mathbf{u}\|_s\big) \|\mathbf{u}\|_s^2. \tag{4.20}$$

The last term of (4.16) is left to estimate. For this, we first notice that $D^\alpha F_0$ is the sum of controlled coefficients multiplied by $D^\beta \mathbf{u}$ (possibly multiplied by ε^2 or ε^4 or …) with $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$ such that $\beta \leqslant \alpha$ (i.e. $\beta_1 \leqslant \alpha_1$ and $\beta_2 \leqslant \alpha_2$). Hence

$$\left| \int D^\alpha F_0 \cdot D^\alpha \mathbf{u} \, d\mathbf{x} \right| \leqslant c\left(1 + \sum_\beta \big\| D^\beta \mathbf{u} \big\|_0\right) \big\| D^\alpha \mathbf{u} \big\|_0 \leqslant c\big(1 + \|\mathbf{u}\|_s\big) \|\mathbf{u}\|_s. \tag{4.21}$$

Secondly $[D^\alpha, A^1 \frac{\partial}{\partial x_1}]\mathbf{u}$ is the sum of controlled coefficients multiplied by $D^\beta \mathbf{u}$ (possibly multiplied by ε or ε^2 or …) and themselves multiplied by $D^\gamma \mathbf{u}$ with $\beta \leqslant \alpha$, $\gamma \leqslant \alpha$ and $\beta + \gamma \leqslant \alpha + (1,0)$ which implies $|\beta| + |\gamma| \leqslant |\alpha| + 1$. When $|\beta| \leqslant s-1$ and $|\gamma| \leqslant s-1$ since $|\beta| + |\gamma| + 1 \leqslant |\alpha| + 2 < 2s$ we deduce that $D^\beta \mathbf{u} \cdot D^\gamma \mathbf{u} \in L^2(\mathbb{R}^2)$ with $\|D^\beta \mathbf{u} \cdot D^\gamma \mathbf{u}\|_0 \leqslant c\|u\|_s^2$ by a classical calculus inequality that can be for instance found in the appendix of Schochet [32]. When $|\alpha| = s$, $|\beta| = s$ and $|\gamma| = 1$ we have $\sup_{\mathbf{x} \in \mathbb{R}^2} |D^\gamma \mathbf{u}| \leqslant \|\mathbf{u}\|_s$. Then $D^\beta \mathbf{u} \cdot D^\gamma \mathbf{u} \in L^2(\mathbb{R}^2)$. When $|\alpha| = s$, $|\beta| = 1$ and $|\gamma| = s$ $\sup_{\mathbf{x} \in \mathbb{R}^2} |D^\beta \mathbf{u}| \leqslant \|\mathbf{u}\|_s$ and then $D^\beta \mathbf{u} \cdot D^\gamma \mathbf{u} \in L^2(\mathbb{R}^2)$. As the same can be done with $[D^\alpha, A^2 \frac{\partial}{\partial x_2}]\mathbf{u}$, we deduce

$$\begin{aligned} \left| \int \left(-\left[D^\alpha, A^1 \frac{\partial}{\partial x_1} \right] \mathbf{u} - \left[D^\alpha, A^1 \frac{\partial}{\partial x_1} \right] \mathbf{u} \right) \cdot D^\alpha \mathbf{u} \, d\mathbf{x} \right| &\leqslant c\big(1 + \|\mathbf{u}\|_s + \|\mathbf{u}\|_s^2\big) \big\| D^\alpha \mathbf{u} \big\|_0 \\ &\leqslant c\big(1 + \|\mathbf{u}\|_s + \|\mathbf{u}\|_s^2\big) \|\mathbf{u}\|_s. \end{aligned} \tag{4.22}$$

Finally, $[D^\alpha, A^0 \frac{\partial}{\partial t}]\mathbf{u}$ is a sum of controlled coefficients multiplied by $\varepsilon D^\beta \mathbf{u}$ (possibly multiplied by ε or ε^2 or …) and themselves multiplied by $D^\gamma \frac{\partial \mathbf{u}}{\partial t}$ with $\gamma < \alpha$ and $(0,0) < \beta \leqslant \alpha$, that implies $|\beta| > 0$ and $|\beta| + |\gamma| \leqslant |\alpha|$. When $|\beta| \leqslant s-1$ and $|\gamma| \leqslant s-2$, since $|\beta| + |\gamma| + 1 \leqslant |\alpha| + 1 < s + (s-1)$, applying classical calculus inequalities, we obtain $D^\beta \mathbf{u} \cdot D^\gamma \frac{\partial \mathbf{u}}{\partial t} \in L^2(\mathbb{R}^2)$, with $\|D^\beta \mathbf{u} \cdot D^\gamma \frac{\partial \mathbf{u}}{\partial t}\|_0 \leqslant c\|u\|_s^{1/2} \|\frac{\partial \mathbf{u}}{\partial t}\|_{s-2}^{1/2} \|u\|_{s-1}^{1/2} \|\frac{\partial \mathbf{u}}{\partial t}\|_{s-1}^{1/2} \leqslant c\|u\|_s \|\frac{\partial \mathbf{u}}{\partial t}\|_{s-1}$. When $|\alpha| = s$, $|\beta| = s$ and $|\gamma| = 0$ we have $\sup_{\mathbf{x} \in \mathbb{R}^2} |D^\gamma \frac{\partial \mathbf{u}}{\partial t}| \leqslant \|\frac{\partial \mathbf{u}}{\partial t}\|_{s-1}$. Then $D^\beta \mathbf{u} \cdot D^\gamma \frac{\partial \mathbf{u}}{\partial t} \in L^2(\mathbb{R}^2)$. When $|\alpha| = s$, $|\beta| = 1$ and $|\gamma| = s-1$, we get $\sup_{\mathbf{x} \in \mathbb{R}^2} |D^\beta \mathbf{u}| \leqslant \|\mathbf{u}\|_s$ and $D^\beta \mathbf{u} \cdot D^\gamma \frac{\partial \mathbf{u}}{\partial t} \in L^2(\mathbb{R}^2)$. Hence, we deduce

$$\left| \int \left(\left[D^\alpha, A^0 \frac{\partial}{\partial t} \right] \mathbf{u} \right) \cdot D^\alpha \mathbf{u} \, d\mathbf{x} \right| \leqslant c\left(1 + \varepsilon \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|_{s-1} \|\mathbf{u}\|_s \right) \|\mathbf{u}\|_s. \tag{4.23}$$

Using inequalities (4.17)–(4.23) and summing (4.16) for $\alpha \leqslant s$, we obtain

$$\sum_{|\alpha| \leqslant s} \left| \frac{d(\int A^0 D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{u} \, d\mathbf{x})}{dt} \right| \leqslant g_1\left(\|\mathbf{u}\|_s, \varepsilon \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|_{s-1} \right), \tag{4.24}$$

for a function g_1 not depending on ε.

Derivating system (4.10) with respect to t, we get

$$\begin{aligned} &A^0 \frac{\partial (D^\alpha \frac{\partial \mathbf{u}}{\partial t})}{\partial t} + A^1 \frac{\partial (D^\alpha \frac{\partial \mathbf{u}}{\partial t})}{\partial x_1} + A^2 \frac{\partial (D^\alpha \frac{\partial \mathbf{u}}{\partial t})}{\partial x_2} + \frac{1}{\varepsilon} S^1 \frac{\partial (D^\alpha \frac{\partial \mathbf{u}}{\partial t})}{\partial x_1} + \frac{1}{\varepsilon} S^2 \frac{\partial (D^\alpha \frac{\partial \mathbf{u}}{\partial t})}{\partial x_2} + \frac{1}{\varepsilon} \left(D^\alpha \frac{\partial \mathbf{u}}{\partial t} \right)^\perp \\ &\quad = -\frac{dA^0}{dt} D^\alpha \frac{\partial \mathbf{u}}{\partial t} - \frac{dA^1}{dt} D^\alpha \frac{\partial \mathbf{u}}{\partial x_1} - \frac{dA^2}{dt} D^\alpha \frac{\partial \mathbf{u}}{\partial x_2} + \frac{dF^\alpha}{dt}. \end{aligned} \tag{4.25}$$

Every previously established estimate remains valid. Moreover, we can set that the entries A^i_{kl} of A^i for $i=1,2$ satisfy

$$\left|\frac{d(A^i_{kl}(t,\frac{t}{\varepsilon},\mathbf{x},\varepsilon\mathbf{u}))}{dt}\right| \leqslant c\left(\frac{1}{\varepsilon}+\varepsilon\|\mathbf{u}\|_s+\varepsilon\left\|\frac{\partial\mathbf{u}}{\partial t}\right\|_{s-1}\right), \tag{4.26}$$

and using, when it is necessary, the same classical calculus procedure as above, we can show that $\frac{dF^\alpha}{dt}$ is in $L^2(\mathbb{R}^2)$ with

$$\left\|\frac{dF^\alpha}{dt}\right\|_0 \leqslant c\left(\frac{1}{\varepsilon}+1+\varepsilon\|\mathbf{u}\|_s+\varepsilon\|\mathbf{u}\|_s^2+\varepsilon\left\|\frac{\partial\mathbf{u}}{\partial t}\right\|_{s-1}+\varepsilon\left\|\frac{\partial\mathbf{u}}{\partial t}\right\|_{s-1}^2\right)\|\mathbf{u}\|_s. \tag{4.27}$$

Then, multiplying Eq. (4.25) by $2\varepsilon D^\alpha(\varepsilon\frac{\partial\mathbf{u}}{\partial t})$ and following the previous method, we obtain

$$\sum_{|\alpha|\leqslant s-1}\left|\frac{d(\int A^0 D^\alpha(\varepsilon\frac{\partial\mathbf{u}}{\partial t})\cdot D^\alpha(\varepsilon\frac{\partial\mathbf{u}}{\partial t})\,d\mathbf{x})}{dt}\right| \leqslant g_2\left(\|\mathbf{u}\|_s,\varepsilon\left\|\frac{\partial\mathbf{u}}{\partial t}\right\|_{s-1}\right), \tag{4.28}$$

for a function g_2 not depending on ε.

As a conclusion, estimates (4.24)–(4.28) together with the fact that

$$\left(\sum_{|\alpha|\leqslant s}\int A^0 D^\alpha\mathbf{u}\cdot D^\alpha\mathbf{u}\,d\mathbf{x}\right)^{1/2} \tag{4.29}$$

is a norm equivalent to $\|\ \|_s$, allows us to set that

$$\frac{d(\|\mathbf{u}\|_s+\varepsilon\|\frac{\partial\mathbf{u}}{\partial t}\|_{s-1})}{dt} \leqslant g\left(\|\mathbf{u}\|_s+\varepsilon\left\|\frac{\partial\mathbf{u}}{\partial t}\right\|_{s-1}\right) \tag{4.30}$$

for a function g not depending on ε and then that the time interval on which the classical solution of (2.5), (2.6) and (2.8) exits does not depend on ε.

Finally, estimate (2.9) is a direct consequence of (4.30). This ends the proof of Theorem 2.1.

5. Asymptotic behavior: proof of Theorem 2.2

In order to deduce the asymptotic behavior as ε goes to 0 of $(\mathcal{I},\mathbf{N})$ we use the method, developed in Tartar [37], Frénod [9] and Frénod and Hamdache [11] and used in Frénod and Sonnendrücker [13,14], which consists in setting a weak formulation with oscillating test functions of system (2.5)–(2.6) or of its equivalent form (4.10). Passing then to the limit using the two scale convergence allows us to set a constraint equation. This constraint equation imposes a form to $(\mathcal{I},\mathbf{N})$. Using test functions satisfying the constraint equation allows us finally to deduce system (2.11).

We start by recalling the following notions linked to two-scale convergence, presented in details in N'Guetseng [26], Allaire [4] and Frénod, Raviart and Sonnendrücker [12]. Let X be a Banach space and let $q\in[1,\infty)$; we denote by X' the dual space of X, $\langle,\rangle$ the duality bracket between X' and X and q' the conjugate exponent of q, such that $\frac{1}{q}+\frac{1}{q'}=1$. We denote by $C_\sharp(\mathbb{R};X)$ the space of continuous 1-periodic functions on $\mathbb{R}$, with values in X. Then given a sequence $(\mathbf{u}(t))$ of functions of $L^{q'}(0,T;X')$ depending on a small parameter ε and a function $\mathbf{U}=\mathbf{U}(t,\theta)$ in $L^{q'}((0,T)\times(0,1);X')=L^{q'}((0,T);L^{q'}(0,1;X'))$, we say that

$$\mathbf{u}\text{ two scale converges to }\mathbf{U}\quad\text{when }\varepsilon\to 0, \tag{5.1}$$

if, for any function $\psi \in L^q(0,T;C_\sharp(\mathbb{R};X))$, we have

$$\lim_{\varepsilon\to 0}\int_0^T \left\langle \mathbf{u}(t), \psi\left(t, \frac{t}{\varepsilon}\right)\right\rangle dt = \int_0^T\int_0^1 \langle \mathbf{U}(t,\theta), \psi(t,\theta)\rangle \,d\theta\, dt. \tag{5.2}$$

We have the following theorem.

Theorem 5.1. *Given a sequence* $(\mathbf{u})$ *depending on a small parameter* ε*, bounded in* $L^{q'}(0,T;X')$*, there exists an extracted subsequence (denoted in the same way) and a function* $\mathbf{U} \in L^{q'}((0,T)\times(0,1);X')$ *such that, when* $\varepsilon \to 0$*,*

$$\mathbf{u}\ \textit{two scale converges to}\ \mathbf{U}, \tag{5.3}$$

$$\mathbf{u}\ \textit{weak-}{*}\ \textit{converges to}\ \int_0^1 \mathbf{U}\,d\theta\ \textit{in}\ L^{q'}\big(0,T;X'\big). \tag{5.4}$$

Having this result at hand, estimate (2.9) yields the two-scale convergence of $\mathbf{u} = (\tilde{\mathcal{I}}, \tilde{\mathbf{N}})$ to $\mathbf{U} = (\mathcal{I}, \mathbf{N}) \in L^\infty((0,T); L^\infty(0,1;(H^s(\mathbb{R}^2))^3))$, up to a subsequence, as ε goes to 0.

Multiplying symmetric hyperbolic system (4.9) by oscillating test functions $\Psi(t, \frac{t}{\varepsilon}, \mathbf{x})$ with functions $\Psi(t,\theta,\mathbf{x})$ being regular, $\mathbb{R}^3$-valued, and such that $\theta \mapsto \Psi(t,\theta,\mathbf{x})$ is 1-periodic and $(t,\mathbf{x}) \mapsto \Psi(t,\theta,\mathbf{x})$ is with compact support in $[0,T)\times\mathbb{R}^2$, and integrating yields

$$\begin{aligned} &-\int_0^T\int_{\mathbb{R}^2} \mathbf{u}\cdot\left(\frac{\partial A^0\Psi}{\partial t} + \frac{1}{\varepsilon}A^0\frac{\partial\Psi}{\partial\theta} + \frac{1}{\varepsilon}\frac{\partial A^0}{\partial\theta}\Psi + \frac{\partial A^1\Psi}{\partial x_1} + \frac{\partial A^2\Psi}{\partial x_2} + \frac{1}{\varepsilon}S^1\frac{\partial\Psi}{\partial x_1} + \frac{1}{\varepsilon}S^2\frac{\partial\Psi}{\partial x_2} + \frac{1}{\varepsilon}\Psi^\perp\right) dt\, d\mathbf{x} \\ &\quad = \int_0^T\int_{\mathbb{R}^2} A^0F\cdot\Psi\, dt\, d\mathbf{x} + \int_{\mathbb{R}^2} \mathbf{u}_0\cdot A^0\Psi(0,0,\cdot)\, d\mathbf{x}. \end{aligned} \tag{5.5}$$

Multiplying (5.5) by ε and passing to the limit yields, since A^0 two-scale converges to I and $\partial A^0/\partial\theta$ to 0,

$$-\int_0^T\int_{\mathbb{R}^2}\int_0^1 \mathbf{U}\cdot\left(\frac{\partial\Psi}{\partial\theta} + S^1\frac{\partial\Psi}{\partial x_1} + S^2\frac{\partial\Psi}{\partial x_2} + \Psi^\perp\right) dt\, d\mathbf{x}\, d\theta = 0, \tag{5.6}$$

which is the weak formulation of

$$\frac{\partial\mathbf{U}}{\partial\theta} + S^1\frac{\partial\mathbf{U}}{\partial x_1} + S^2\frac{\partial\mathbf{U}}{\partial x_2} + \mathbf{U}^\perp = 0, \tag{5.7}$$

or

$$\frac{\partial\mathcal{I}}{\partial\theta} + \nabla\cdot\mathbf{N} = 0, \qquad \frac{\partial\mathbf{N}}{\partial\theta} + \mathbf{N}^\perp + \nabla\mathcal{I} = 0. \tag{5.8}$$

For $\mathcal{S}'$ being the dual space of the space of infinitely differentiable functions with fast decay we consider, for $k \in 2\pi\mathbb{Z}$ and $l = (l_1, l_2) \in \mathbb{R}^2$, $(\hat{\mathcal{I}}, \hat{\mathbf{N}}) = (\hat{\mathcal{I}}(k, l_1, l_2), \hat{\mathbf{N}}(k, l_1, l_2))$ the Fourier transform in $\mathcal{S}'$ of $(\mathcal{I}, \mathbf{N})$. From Eq. (5.10), we deduce that $(\hat{\mathcal{I}}(k, l_1, l_2), \hat{\mathbf{N}}(k, l_1, l_2))$ is the solution to

$$\begin{aligned} &k\hat{\mathcal{I}} + l_1\hat{\mathbf{N}}_1 + l_2\hat{\mathbf{N}}_2 = 0, \\ &k\hat{\mathbf{N}}_1 - \hat{\mathbf{N}}_2 + l_1\hat{\mathcal{I}} = 0, \\ &k\hat{\mathbf{N}}_2 + \hat{\mathbf{N}}_1 + l_2\hat{\mathcal{I}} = 0. \end{aligned} \tag{5.9}$$

Since the determinant of this system is $k(l_1^2 + l_2^2 - k^2 - 1)$, it has non zero solutions if $l_1^2 + l_2^2 = k^2 + 1$, or if $k = 0$. Hence any non zero solution of (5.8) is made of two terms. The first of those terms has a Fourier transform supported on the set $\{(k, l_1, l_2) \in \mathbb{R}^3,\ k \in 2\pi\mathbb{Z},\ l_1^2 + l_2^2 = k^2 + 1\}$. As a function being the Fourier transform of a function in $L^\infty(0, 1; (H^s(\mathbb{R}^2))^3)$ having a support in such set can only be 0, we deduce that this first term is 0. The second term has a Fourier transform supported on the set $\{(k, l_1, l_2) \in \mathbb{R}^3,\ k = 0\}$, then it does not depend on θ. Hence, we may conclude that $\mathbf{U} = (\mathcal{I}, \mathbf{N})$ does not depend on θ. As a consequence $\mathbf{U} = (\mathcal{I}, \mathbf{N})$ is also the weak-$*$ limit of $\mathbf{u} = (\tilde{\mathcal{I}}, \tilde{\mathbf{N}})$ and is solution to

$$\nabla \cdot \mathbf{N} = 0, \qquad \mathbf{N}^\perp + \nabla\mathcal{I} = 0. \tag{5.10}$$

From this constraint equation, we deduce the form of $(\mathcal{I}, \mathbf{N})$ given by (2.10).

For any regular function φ we define the test function Ψ satisfying the constraint equation by

$$\Psi_1(t, \mathbf{x}) = \varphi(t, \mathbf{x}), \qquad \Psi_2(t, \mathbf{x}) = -\frac{\partial\varphi}{\partial x_2}(t, \mathbf{x}), \qquad \Psi_3(t, \mathbf{x}) = \frac{\partial\varphi}{\partial x_1}(t, \mathbf{x}). \tag{5.11}$$

Using this function in (5.5) cancels terms containing $1/\varepsilon$ factors. Since A^0 is a regular oscillating function two-scale converging to I, $1/\varepsilon\, \partial A^0_{11}/\partial\theta$ a regular oscillating function two-scale converging to $\partial\mathcal{H}/\partial\theta$, A^1 a regular oscillating function two-scale converging to $\mathbf{M}_1 I$ and A^2 a regular oscillating function two-scale converging to $\mathbf{M}_2 I$, passing to the limit yields

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^2} \int_0^1 \mathbf{U} \cdot \left(\frac{\partial\Psi}{\partial t} + \frac{\partial\mathcal{H}}{\partial\theta} \begin{pmatrix} \Psi_1 \\ 0 \\ 0 \end{pmatrix} + \frac{\partial \mathbf{M}_1\Psi}{\partial x_1} + \frac{\partial \mathbf{M}_2\Psi}{\partial x_2} \right) dt\, d\mathbf{x}\, d\theta \\ &\quad = \int_0^T \int_{\mathbb{R}^2} \int_0^1 F \cdot \Psi\, dt\, d\mathbf{x}\, d\theta + \int_{\mathbb{R}^2} \int_0^1 \mathbf{u}_0 \cdot \Psi(0, 0, \cdot)\, d\mathbf{x}\, d\theta. \end{aligned} \tag{5.12}$$

Since, neither $\mathbf{U}$ nor Ψ depends on θ, this last equation gives

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^2} \mathbf{U} \cdot \left(\frac{\partial\Psi}{\partial t} + \frac{\partial(\int_0^1 \mathbf{M}_1\, d\theta)\Psi}{\partial x_1} + \frac{\partial(\int_0^1 \mathbf{M}_2\, d\theta)\Psi}{\partial x_2} \right) dt\, d\mathbf{x} \\ &\quad = \int_0^T \int_{\mathbb{R}^2} \int_0^1 F\, d\theta \cdot \Psi\, dt\, d\mathbf{x} + \int_{\mathbb{R}^2} \mathbf{u}_0 \cdot \Psi(0, \cdot)\, d\mathbf{x}, \end{aligned} \tag{5.13}$$

or, using expressions of $\mathbf{U}$, Ψ and F,

$$
\begin{aligned}
&-\int_0^T\int_{\mathbb{R}^2}\mathcal{I}\left(\frac{\partial\varphi}{\partial t}+\frac{\partial(\int_0^1\mathbf{M}_1\,d\theta)\varphi}{\partial x_1}+\frac{\partial(\int_0^1\mathbf{M}_2\,d\theta)\varphi}{\partial x_2}\right)\\
&\quad+\frac{\partial\mathcal{I}}{\partial x_2}\left(\frac{\partial\frac{\partial\varphi}{\partial x_2}}{\partial t}+\frac{\partial(\int_0^1\mathbf{M}_1\,d\theta)\frac{\partial\varphi}{\partial x_2}}{\partial x_1}+\frac{\partial(\int_0^1\mathbf{M}_2\,d\theta)\frac{\partial\varphi}{\partial x_2}}{\partial x_2}\right)\\
&\quad+\frac{\partial\mathcal{I}}{\partial x_1}\left(\frac{\partial\frac{\partial\varphi}{\partial x_1}}{\partial t}+\frac{\partial(\int_0^1\mathbf{M}_1\,d\theta)\frac{\partial\varphi}{\partial x_1}}{\partial x_1}+\frac{\partial(\int_0^1\mathbf{M}_2\,d\theta)\frac{\partial\varphi}{\partial x_1}}{\partial x_2}\right)dt\,d\mathbf{x}\\
&=\int_0^T\int_{\mathbb{R}^2}-\left(\frac{\partial(\int_0^1\mathcal{H}\,d\theta)}{\partial x_1}\left(-\frac{\partial\mathcal{I}}{\partial x_2}\right)+\frac{\partial(\int_0^1\mathcal{H}d\theta)}{\partial x_2}\left(\frac{\partial\mathcal{I}}{\partial x_1}\right)+\left(\frac{\partial(\int_0^1\mathbf{M}_1\,d\theta)}{\partial x_1}+\frac{\partial(\int_0^1\mathbf{M}_2\,d\theta)}{\partial x_2}\right)\mathcal{I}\right)\varphi\\
&\quad-\left(\int_0^1\mathbf{W}_1\,d\theta-\left(\frac{\partial(\int_0^1\mathbf{M}_1\,d\theta)}{\partial x_1}\left(-\frac{\partial\mathcal{I}}{\partial x_2}\right)+\frac{\partial(\int_0^1\mathbf{M}_1\,d\theta)}{\partial x_2}\left(\frac{\partial\mathcal{I}}{\partial x_1}\right)\right)\right)\frac{\partial\varphi}{\partial x_2}\\
&\quad+\left(\int_0^1\mathbf{W}_2\,d\theta-\left(\frac{\partial(\int_0^1\mathbf{M}_2\,d\theta)}{\partial x_1}\left(-\frac{\partial\mathcal{I}}{\partial x_2}\right)+\frac{\partial(\int_0^1\mathbf{M}_2\,d\theta)}{\partial x_2}\left(\frac{\partial\mathcal{I}}{\partial x_1}\right)\right)\right)\frac{\partial\varphi}{\partial x_1}\,dt\,d\mathbf{x}\\
&\quad+\int_{\mathbb{R}^2}\tilde{\mathcal{I}}_0\varphi-(\tilde{\mathbf{N}}_0)_1\frac{\partial\varphi}{\partial x_2}+(\tilde{\mathbf{N}}_0)_2\frac{\partial\varphi}{\partial x_1}\,d\mathbf{x}.
\end{aligned}\tag{5.14}
$$

We have here a weak formulation of (2.11). Since this equation is linear, it is easy to show that its solution is unique. From this, we can finally deduce that the whole sequence $\mathbf{u}$ weak-$*$ converges to $\mathbf{U}$ as $\varepsilon \to 0$, ending the proof.

6. Conclusion and perspectives

In this paper, we set out equations modeling the long term evolution of the perturbation $\tilde{\mathcal{I}}$ of the ocean free surface elevation and of the perturbation $\tilde{\mathbf{N}}$ of the velocity field. Because of the tide wave, those models contain and generate oscillations with high frequency. If numerical simulations of near coastal ocean waters are needed, directly using those models could be very expensive because of the oscillations. Nevertheless, the result given in Theorem 2.2, which says that $(\tilde{\mathcal{I}}, \tilde{\mathbf{N}})$ weak-$*$ converges to $(\mathcal{I}, \mathbf{N})$, suggests a way to use those models for numerical simulations of near coastal ocean waters. As an intuitive interpretation of it we could say that

$$
\tilde{\mathcal{I}}(t,\mathbf{x}) \text{ is close to } \mathcal{I}(t,\mathbf{x}) \quad \text{and} \quad \tilde{\mathbf{N}}(t,\mathbf{x}) \text{ is close to } \mathbf{N}(t,\mathbf{x}). \tag{6.1}
$$

Hence, since Eqs. (2.11)–(2.12) neither contain nor generate oscillations with frequency $1/\varepsilon$, we can solve it using a numerical method involving a time step which does not need to be small compared with ε. Hence solving (2.11)–(2.12) and then reconstructing $(\mathcal{I}, \mathbf{N})$ via (2.10) in place of solving (2.5)–(2.6)–(2.8) directly may give good results in a shorter computational time.

Among the tasks listed at the beginning of this paper, we realized significant steps in the direction of building numerical methods in our previous paper [2] and in coastal ocean modeling over long time periods in the present one.

The next step, we shall provide in a forthcoming paper, will consist in using the models set out here in order to compute the ocean fields in a real near coastal ocean area and to couple this to the numerical method proposed in [2]. This will allow us to make forecasts in a real coastal ocean region.

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